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Wavefunction and energy level formula for two charged particles with magnetic interaction*

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Abstract
We derive the wavefunction and energy level formula for two charged particles with magnetic interaction, i.e., the Hamiltonian includes both two-body Coulomb interaction and a kinetic coupling. It is by virtue of the EPR entangled state representation we can conveniently derive the exact result.

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1. Introduction

Solving Schrödinger equations for given Hamiltonians is always a challenge to theoretical physicists, since only very limited dynamic models can be solved analytically. Choosing a suitable quantum-mechanical representation may bring much convenience for working out exact solutions of Schrödinger equations [1]. In this work we begin by considering an electron located above an infinite dielectric medium plane at distance x, the dielectric constant is ϵ. According to the electric-mirror imaging method in electrodynamics the static electric potential is

$$V(x) = \begin{cases} -g/x & g = \frac{e^2}{4\pi\epsilon_0}\frac{1}{4}\frac{\epsilon-1}{\epsilon+1} > 0 \qquad x > 0 \\ \infty & \qquad\qquad\qquad\qquad\qquad x < 0 \end{cases} \tag{1}$$

where ϵ_0 is the dielectric constant for the vacuum. Now we extend (1) to the following bipartite case,

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{g}{X_1 - X_2} + kP_1P_2 \tag{2}$$

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where the terms $\frac{g}{X_1-X_2}$ and kP_1P_2 denote a Coulomb potential and a kinetic coupling, respectively. (The Hamiltonian (2) and later (3) apply to a system composed of two separate charged particles moving in free space including the effects of their interaction via the vector potential $\vec{j}\cdot\vec{A}$ (the current of one particle times the vector potential of the other particle) where $\vec{j}$ is the current and $\vec{A}$ is the electromagnetic potential). In the nonrelativistic case, the complication arising from the Lienard–Wiechert potential can be neglected. Aside from the dependence on their separation, such a term is proportional to the dot product of the two momenta (magnetic interaction). So if the distance between the two particles is much larger than the amount that they move over the time of interest, then the distance can be approximated by a constant and all of the dynamics depends only on the dot product of the momenta. So far as we know, the wavefunction and energy eigenvalues for (2) have not been reported in the literature before, not to mention the generalization of (2) to the three-dimensional case with the Hamiltonian being

$$H=\frac{\vec{P}_1^2}{2m_1}+\frac{\vec{P}_2^2}{2m_2}+k\vec{P}_1\cdot\vec{P}_2+\frac{g}{|\vec{R}_1-\vec{R}_2|}=\sum_{i=1}^{3}\left(\frac{P_{1i}^2}{2m_1}+\frac{P_{2i}^2}{2m_2}+kP_{1i}P_{2i}\right)+\frac{g}{|\vec{R}_1-\vec{R}_2|}$$
$$\text{where}\quad \vec{R}_1=(X_1,Y_1,Z_1)\qquad \vec{R}_2=(X_2,Y_2,Z_2). \tag{3}$$

The purpose of this work is to find the solutions for (2) and (3). We would expect that the energy spectrum of H can be divided into two parts: a continuous part corresponding to the motion of the centre of the mass, and a discrete one responsible for the relative motion of two particles. We shall employ the entangled state representation of continuum variables to explicitly derive the energy formulae. The concept of quantum entanglement regarding two particles' relative coordinate $X_r=X_1-X_2$ and total momentum $P=P_1+P_2$ was initiated by Einstein Podolsky and Rosen (EPR) in [2], to criticize the incompleteness of quantum mechanics. Based on the commutator $[X_r,P]=0$, we can establish the common eigenvector $|\eta\rangle$ of X_r and P, which is qualified to be a quantum-mechanical representation. Since the term $\frac{g}{X_1-X_2}$ in (2) is diagonalized in the $\langle\eta|$ representation, we are naturally led to working out the solution of (2) in the entangled state representation. At this point, we mention that there is no direct relevance of the original EPR argument of quantum entanglement for solving the eigenvalue problems. This is because EPR's original idea of entanglement is involved in certain two-particle states with the property that a measurement of one chosen variable of particle 1 completely determines the outcome of a measurement of the corresponding variable of particle 2. At the time of measurement, the two particles may be so far apart that no influence resulting from one measurement can possibly propagate to the other particle in the available time. In contrast, the two particles described by the Hamiltonian (2) contain mutual interaction, so the EPR argument cannot be applied to this system. In this sense, the usefulness of EPR pairs we shall present in the following sections is merely accidental, despite some superficial formal analogies.

2. Brief review of the EPR entangled states

Since

$$[X_r,P]=0 \tag{4}$$

they can possess the common eigenvector $|\eta\rangle$ [3],

$$|\eta\rangle=\exp\left[-\frac{|\eta|^2}{2}+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\right]|00\rangle\qquad \eta=\eta_1+\mathrm{i}\eta_2 \tag{5}$$

where the Bose creation operator $a_i^\dagger$ and the annihilation operator a_i are related to X_i and P_i by

$$X_i = \frac{a_i + a_i^\dagger}{\sqrt{2}} \qquad P_i = \frac{a_i - a_i^\dagger}{\sqrt{2}i} \qquad (\hbar = 1) \tag{6}$$

and $\left[a_i, a_j^\dagger\right] = \delta_{ij}$. It has been proved in [3] that

$$\begin{aligned} &\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta|\eta\rangle \qquad &&\left(a_2 - a_1^\dagger\right)|\eta\rangle = -\eta^*|\eta\rangle \\ &X_r|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle \qquad &&P|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle. \end{aligned} \tag{7}$$

Moreover, the $|\eta\rangle$ states span an orthonormal and complete set, i.e.,

$$\langle\eta|\eta'\rangle = \pi\delta(\eta - \eta')\delta(\eta^* - \eta'^*) \tag{8}$$

and

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1. \tag{9}$$

Hence $|\eta\rangle$ is qualified to be a quantum-mechanical representation for bipartite systems. In [4], we have used $|\eta\rangle$ to further discuss the correlative amplitude–operational phase entanglement. According to [5] we can derive the Schmidt decomposition of $|\eta\rangle$ [6],

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2}\int_{-\infty}^{\infty}\mathrm{d}p|p + \sqrt{2}\eta_2\rangle_1 \otimes |-p\rangle_2\,\mathrm{e}^{-\mathrm{i}\sqrt{2}\eta_1 p} \tag{10}$$

where $|p\rangle_i$ is the momentum eigenstate. We shall deal with Hamiltonian (2) in the $|\eta\rangle$ representation.

3. The energy eigenvalues and wavefunctions for Hamiltonian (2)

In equation (2) when $k = 0$, one can easily separate the motion of the centre-of-mass from the relative motion of the two particles. By introducing

$$\mu_1 = \frac{m_1}{M} \qquad \mu_2 = \frac{m_2}{M} \qquad M = m_1 + m_2 \qquad \mu = \frac{m_2 m_1}{M} \qquad P_r = \mu_2 P_1 - \mu_1 P_2 \tag{11}$$

the Hamiltonian (2) is split into

$$H = H_1 + H_2 \qquad H_1 = \frac{P^2}{2M} \qquad H_2 = \frac{P_r^2}{2\mu} + \frac{g}{X_r}. \tag{12}$$

However, when $k \neq 0$, the Hamiltonian becomes

$$H = \left(\frac{1}{2M} + k\mu_1\mu_2\right)P^2 + \left(\frac{1}{2\mu} - k\right)P_r^2 + k(\mu_2 - \mu_1)PP_r + \frac{g}{X_r} \tag{13}$$

the existence of $k(\mu_2 - \mu_1)PP_r$ prevents one from separating out the centre-of-mass motion. Then a question naturally arises: how to solve the eigenstate problem $H|E_n\rangle = E_n|E_n\rangle$?

An effective way of converting the eigenstate problem into a c-number differential equation is by virtue of the entangled state representation $|\eta\rangle$, we project $H|E_n\rangle = E_n|E_n\rangle$ onto the $\langle\eta|$ state,

$$\langle\eta|H|E_n\rangle = E_n\langle\eta|E_n\rangle. \tag{14}$$

From (10) we have

$$P_r|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2}\int_{-\infty}^{\infty}\mathrm{d}p(\mu_2 p + \sqrt{2}\eta_2 + \mu_1 p)|p + \sqrt{2}\eta_2\rangle_1 \otimes |-p\rangle_2\, \mathrm{e}^{-\mathrm{i}\sqrt{2}\eta_1 p}$$
$$= \sqrt{\frac{1}{2}}\left[\mathrm{i}\frac{\partial}{\partial\eta_1} - (\mu_1 - \mu_2)\eta_2\right]|\eta\rangle. \tag{15}$$

Substituting (13), (15) and (7) into (14), we obtain

$$\left\{\left(\frac{1}{M} + 2k\mu_1\mu_2\right)\eta_2^2 + \frac{1}{2}\left(k - \frac{1}{2\mu}\right)\left[\frac{\partial}{\partial\eta_1} - \mathrm{i}(\mu_1 - \mu_2)\eta_2\right]^2 - \mathrm{i}\eta_2 k(\mu_2 - \mu_1)\left[\frac{\partial}{\partial\eta_1} - \mathrm{i}(\mu_1 - \mu_2)\eta_2\right] + \frac{g}{\sqrt{2}\eta_1}\right\}\psi_n(\eta) = E_n\psi_n(\eta) \tag{16}$$

where $\psi_n(\eta) = \langle\eta|E_n\rangle$ is the wavefunction of the eigenstate in the $|\eta\rangle$ representation.

In order to solve equation (16), we first simplify it to the standard form by eliminating the term of the first order of differentiation. This can be done by introducing $\varphi_n = \exp[-\mathrm{i}(\mu_1 - \mu_2)\eta_1\eta_2]\psi_n(\eta)$ and substituting φ_n into (16) yields

$$\left\{\frac{1}{2}\left(k - \frac{1}{2\mu}\right)\frac{\partial^2}{\partial\eta_1^2} - \mathrm{i}\eta_2 k(\mu_2 - \mu_1)\frac{\partial}{\partial\eta_1} + \left[\left(\frac{1}{M} + 2k\mu_1\mu_2\right)\eta_2^2 + \frac{g}{\sqrt{2}\eta_1} - E_n\right]\right\}\varphi_n(\eta) = 0. \tag{17}$$

We further introduce

$$\varphi_n'(\eta_1, \eta_2) = \mathrm{e}^{-\mathrm{i}\eta_1\rho}\varphi_n \qquad \rho \equiv 2\eta_2 k\mu(\mu_1 - \mu_2)/(1 - 2k\mu) \tag{18}$$

into (17), this gives rise to

$$\left\{\frac{1}{2}\left(k - \frac{1}{2\mu}\right)\frac{\partial^2}{\partial\eta_1^2} + \frac{g}{\sqrt{2}\eta_1} - E_n + \frac{1 - k^2\mu M}{M(1 - 2\mu k)}\eta_2^2\right\}\varphi_n'(\eta) = 0. \tag{19}$$

Then we perform a Fourier transform on $\varphi_n'(\eta_1, \eta_2)$ with respect to the variable η_1 only,

$$\varphi_n'(\eta_1, \eta_2) \longleftrightarrow \phi_n'(\xi_1, \eta_2) = \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi_1\eta_1}\varphi_n'(\eta_1, \eta_2)\,\mathrm{d}\eta_1. \tag{20}$$

It then follows $\frac{\mathrm{d}}{\mathrm{d}\eta_1}\varphi_n'(\eta_1, \eta_2) \longleftrightarrow -\mathrm{i}\xi_1\phi_n'(\xi_1, \eta_2)$ and

$$\frac{1}{\eta_1}\varphi_n'(\eta_1, \eta_2) \longleftrightarrow \mathrm{i}\int_{-\infty}^{\xi_1}\phi_n'(\xi_1', \eta_2)\,\mathrm{d}\xi_1', \tag{21}$$

which is analogous to the Fourier transform between coordinate representation $\langle x_1|$ and the momentum representation $\langle p_1|$ that while $\langle x_1|\frac{1}{X_1} = \frac{1}{x_1}\langle x_1|$, $\langle x_1|P_1 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x_1}\langle x_1|$, the inverse of coordinate operator X_1 in $\langle p_1|$ representation is expressed as $\langle p_1|\frac{1}{X_1} = -\mathrm{i}\int_{-\infty}^{p_1}\mathrm{d}p_1'\langle p_1'|$, since $\langle p_1|X_1 = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p_1}\langle p_1|$. Now the differential equation (19) becomes

$$\frac{\mathrm{i}g}{\sqrt{2}}\int_{-\infty}^{\xi_1}\phi_n'(\xi_1', \eta_2)\,\mathrm{d}\xi_1' + \left[\frac{1}{2}\left(\frac{1}{2\mu} - k\right)\xi_1^2 + \frac{1 - k^2\mu M}{M(1 - 2\mu k)}\eta_2^2 - E_n\right]\phi_n'(\xi_1, \eta_2) = 0. \tag{22}$$

This integral-differential equation is only issued on ξ_1, while η_2 appears only as a parameter, so in the following we do not write it explicitly. Equation (22) can be rewritten as

$$\frac{\phi_n'(\xi_1)}{\int_{-\infty}^{\xi_1}\phi_n'(\xi_1')\,\mathrm{d}\xi_1'} = -\frac{\sqrt{2}\mathrm{i}g}{\left(\frac{1}{2\mu} - k\right)}\frac{1}{\xi_1^2 + f^2} \tag{23}$$

where

$$f^2 = \frac{4\mu}{1-2\mu k}\left(\frac{1-k^2\mu M}{M(1-2\mu k)}\eta_2^2 - E_n\right) \qquad \text{since} \quad E_n < 0 \quad f^2 > 0. \tag{24}$$

Solving equation (23) we obtain

$$\int_{-\infty}^{\xi_1} \phi_n'(\xi_1')\,\mathrm{d}\xi_1' = \exp\left[\frac{-\mathrm{i}\sqrt{2}g}{\left(\frac{1}{2\mu}-k\right)f}\arctan\frac{\xi_1}{f}\right] + C. \tag{25}$$

At this point we note that $\arctan\frac{\xi_1}{f}$ is a multi-valued function. In order that the wavefunction is unique, we should require $\frac{\sqrt{2}g}{\left(\frac{1}{2\mu}-k\right)f} = 2n$, n is an integer. Then

$$f = \frac{g}{\sqrt{2}n\left(\frac{1}{2\mu}-k\right)}. \tag{26}$$

This leads to the energy eigenvalues,

$$E_n = \frac{1-k^2\mu M}{M(1-2\mu k)}\eta_2^2 - \frac{1}{2}\left(\frac{1}{2\mu}-k\right)f^2 = \frac{1-k^2\mu M}{M(1-2\mu k)}\eta_2^2 - \frac{g^2\mu}{2n^2\hbar^2(1-2k\mu)} \tag{27}$$

where we have recovered $\hbar$, the energy in the first term is continuous and related to the centre-of-mass motion, while the second term is quantized, in agreement with our early expectation.

Or we can rewrite (27) as

$$E_n = \frac{1-k^2 m_1 m_2}{(1-2\mu k)}\frac{P^2}{2M} - \frac{g^2\mu}{2n^2\hbar^2(1-2k\mu)}$$

since the total momentum $P = P_1 + P_2$ commutes with the Hamiltonian. We can see that when $k = 0$, the energy levels return to the familiar case $E_n = \frac{P^2}{2M} - \frac{g^2\mu}{2n^2\hbar^2}$. From equation (27) we also note that the presence of kinematic coupling increases the separation of two adjacent energy levels for $k > 0$.

Differentiating the two sides of (25) with respect to ξ_1, we obtain the wavefunction

$$\phi_n'(\xi_1) = \frac{-2\sqrt{2}\mu g\mathrm{i}}{(1-2\mu k)}\frac{1}{\left(\frac{\sqrt{2}\mu g}{n(1-2\mu k)}\right)^2 + \xi_1^2}\exp\left[-2n\mathrm{i}\arctan\frac{n(1-2\mu k)\xi_1}{\sqrt{2}\mu g}\right]. \tag{28}$$

4. The energy eigenvalues and wavefunctions for Hamiltonian (3)

We now consider the Hamiltonian (3) which is a three-dimensional generalization of (2). Following the same procedures as from (14) to (19) we have

$$\left\{\frac{1}{2}\left(k-\frac{1}{2\mu}\right)\nabla_{\vec\eta_1}^2 + \frac{g}{\sqrt{2}|\vec\eta_1|} - E_n + \frac{1-k^2\mu M}{M(1-2\mu k)}\vec\eta_2^2\right\}\varphi_n'(\vec\eta) = 0 \tag{29}$$

where

$$\vec\eta = (\alpha,\beta,\gamma) \qquad \vec\eta_1 = (\alpha_1,\beta_1,\gamma_1) \qquad \vec\eta_2 = (\alpha_2,\beta_2,\gamma_2)$$

$$\nabla_{\vec\eta_1}^2 = \frac{\partial^2}{\partial\alpha_1^2} + \frac{\partial^2}{\partial\beta_1^2} + \frac{\partial^2}{\partial\gamma_1^2} \qquad \vec\eta_2^2 = \alpha_2^2+\beta_2^2+\gamma_2^2 \qquad |\vec\eta_1| = \sqrt{\alpha_1^2+\beta_1^2+\gamma_1^2}.$$

Changing $\vec\eta_1 = (\alpha_1,\beta_1,\gamma_1)$ to the spherical polar coordinates (r,θ,ϕ),

$$\alpha_1 = r\sin\theta\sin\phi \qquad \beta_1 = r\sin\theta\cos\phi \qquad \gamma_1 = r\cos\theta$$

(29) becomes

$$\frac{1}{2}\left(k-\frac{1}{2\mu}\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_n'}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi_n'}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2\varphi_n'}{\partial\phi^2}\right]\varphi_n' + \left(\frac{g}{\sqrt{2}r}-E_n+\frac{1-k^2\mu M}{M(1-2\mu k)}\vec{\eta}_2^2\right)\varphi_n'=0. \tag{30}$$

Comparing (30) with the well-known differential equation for the radial Coulomb potential $-\frac{e^2}{r}$,

$$-\frac{1}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]-\frac{e^2}{r}\psi=E\psi \tag{31}$$

we see the correspondence

$$\frac{1}{m}\longleftrightarrow\frac{1}{2\mu}-k \qquad -e^2\longleftrightarrow\frac{g}{\sqrt{2}} \qquad E\longleftrightarrow E_n-\frac{1-k^2\mu M}{M(1-2\mu k)}\vec{\eta}_2^2. \tag{32}$$

Therefore the energy level formula in the three-dimensional case is

$$E_n=\frac{1-k^2\mu M}{M(1-2\mu k)}\vec{\eta}_2^2-\frac{g^2\mu}{2n^2\hbar^2(1-2k\mu)}. \tag{33}$$

In summary, by virtue of the EPR entangled state representation we have derived the wavefunction and energy level formula for two particles which possesses both two-body Coulomb interaction and a kinetic coupling. We emphasize that there is no direct relevance of the original EPR argument of quantum entanglement for solving the eigenvalue problems, i.e., the usefulness of EPR entangled state $|\eta\rangle$ is merely for the convenience of solving the Hamiltonian. This work shows that entangled state representations may simplify calculations for some dynamic problems. Thus finding more multi-partite entangled state representations in quantum mechanics represents a challenge to future investigations solving more general Hamiltonians.

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References

[1] Dirac P A M 1958 *Principles of Quantum Mechanics* 3rd edn (Oxford: Oxford University Press)
[2] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[3] Fan Hong-yi and Klauder J R 1994 *Phys. Rev.* A **49** 704
[4] Fan Hong-yi 2002 *J. Phys. A: Math. Gen.* **35** 1007
Fan Hong-yi and Sun Zhi-hu 2000 *J. Phys. A: Math. Gen.* **33** 4871
Fan Hong-yi and Sun Zhi-hu 2001 *J. Phys. A: Math. Gen.* **34** 1629
[5] Preskill J 1998 *Lectures for Physics 229: Quantum Information and Computation* (Pasadena, CA: California Institute of Technology)
[6] Fan Hong-yi 2002 *Phys. Lett.* A **294** 253